On Howson's Theorem

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Abstract

Let U and V be finitely generated subgroups of a free group. We derive bounds on the rank of $U \cap V$ in terms of the ranks of U, V and the subgroup $U \vee V$ generated by U and V.

1. Introduction

If U and V are finitely generated subgroups of a free group, then $U \cap V$ is also finitely generated. This result is known as Howson's Theorem. Several bounds are known for the rank $r_{U \cap V}$ of $U \cap V$ in terms of the ranks of U and V. In his original paper [2] Howson derived the general upper bound

$$r_{U \cap V} - 1 \leq 2(r_U - 1)(r_V - 1) + r_U + r_V$$

and H. Neumann [4] improved it to

$$(1) \qquad r_{U \cap V} - 1 \leq 2(r_U - 1)(r_v - 1).$$

In she also raised the question whether the factor 2 can be dropped in (1), which is true, for example, if U or V has finite index in $U \vee V$. This problem has become known as the "Hanna Neumann Conjecture".

Subsequently, Burns [1] improved the general upper bound to

$$(1) \qquad r_{U \cap V} - 1 \leq 2(r_U - 1)(r_U - 1) - \min\{r_U - 1, r_V - 1\}.$$

The original proof is rather difficult and several shorter versions have been published, notably one by P. Nickolas **??**.

Since then little progress was achieved until 1990, when G. Tardos [7] showed the validity of the Hanna Neumann Conjecture in the case that at least one of the groups U, V has rank 2. Recently, W. Dicks [**?**] announced a positive solution to the H. Neumann Conjecture.

It should be noted that (1) and (2) remain true if the left-hand side is replaced by the sum

$$S(U, V) = \sum_{G \in \mathcal{G}} (r_G - 1),$$

where $\mathcal{G}$ consists of all nontrivial intersections of the form $U \cap h^{-1}Vh$, where h runs through a system of representatives of the double cosets $V \times U$. The question whether

$$(3) \qquad S(U, V) \leq (r_U - 1)(r_V - 1)$$

has been called the "Strengthended H. Neumann Question" by W. Neumann [**?**]. As Tardos already remarks his proof in fact establishes this stronger version in the case when U or V has rank 2.

The aim of this note is to introduce the rank of the subgroup $U \vee V$ generated by U and V, the no-called join of U and V, into this context. We shall prove the following result.

Theorem Let U, V be finitely generated subgroups of a free group. Then $U \cap V$ is also finitely generated and the following bounds hold:

$$(4) \qquad (r_{U \cap V} - 1)(r_{U \vee V} - 1) \leq (r_U - 1)(r_V - 1)$$

if U or V has finite index in $U \vee V$.

$$(5) \qquad r_{U \cap V} - 1 \leq 2(r_U - 1)(r_V - 1) - (r_{U \vee V} - 1)(\min\{r_U, v_V\} - 1)$$

if there are no free factors U^* and V^* if U and V, respectively, such that $U^* \cap V^* = U \cap V$.

We conjecture that (4) also holds in case U and V are if infinite index, provided that there are no free factors U^* and V^* if U and V, respectively, such that $U^* \cap V^* = U \cap V$.

For the proof of (5) we shall make use of a pushout construction as used by Stallings [6]

To this end we first recall sveral properties of Cayley graphs and their quotients. Let F be a free group, freely generated by S. Then the Cayley graph $\Gamma(F, S)$ of F with respect to S is a tree with vertex set F and (oriented) edges of the form

$$(g, gs), G \in F, s \in S,$$

where g is the origin and gs the terminus of (g, gs). For our purposes it will suffice to let $S = \{a, b\}$. Also, we shall color every edge of the form (g, ga) red and the others blue.

F acts on Γ by left multiplication and for every subgroup H of F the quotient graph Γ/H is a graph of cyclomatic number r_H. The smallest connected subgraph of Γ/H containing all cycles of Γ/H is called the core of Γ/H. If r_H is finite the core graph of Γ/H will thus also be finite.

Morover, any connected graph Δ satisfying the following conditions is a core graph of Γ with respect to some subgroup H:

(a) Every vertex of Δ has degree at least 2.

(b) Every vertex is the origin of at most one edge of every color (i.e. red and blue in our case) and the terminus of at most one edge of every color.

Also, the cyclomatic number of Δ equals the rank of H.

We recall now that U, V are given as subgroups of some free group. To fix ideas, we shall embed $U \vee V$ into F such that the can graph of $\Gamma(U \vee V)$ contain only vertices of degree two or three. Let Δ be this graph.

Then the care graphs Γ_0 of $\Gamma(U \cap V)$, Γ_1 of Γ/U and Γ_2 of Γ/V are all well defined. Since the vertices of $\Gamma_0, \Gamma, \Gamma_2$ and Δ are cosets of $U \cap V, U, V$ and $U \vee V$ in F we can define mappings from Γ_0 to Γ_1, Γ_2 and Δ by inclusion of cosets and similarly from Γ_1 and Γ_1 and Γ_2 to Δ.

These mappings lead to the following commutative diagram of locally injective maps

In this diagram Δ is the pushout and Γ_0 a component of the pullback of Γ_1 and Γ_2.

It is not hard to show (see Stallings [6]) that Δ is a core graph of $U \vee V$ and Γ_0 of $U \cap V$.

Clearly neither Γ_0 nor Γ_1 or Γ_2 can have vertices of degree ¿ 3.

Lemma Suppose α_1 and α_2 of the above diagramm are vertex-surjective. Then every vertex of degree three in Δ has a preimage of degree three in Γ_1 or in Γ_2.

Proof. Let z be a vertex of degree 3 in Δ which has no preimage of degree 3. We can assume without loss of generality that z is the origin of a blue edge, the terminus of a blue edge (these two edges can be identical) and the origin of a red edge.

Then z must be the image of at least two vertices of degree 2 in Γ_1 or Γ_2 of the following type:

(a) origin and terminus of blue edges
(b) origin of a blue and red edge
(c) terminus of a blue and origin of a red edge.

Unless both of these vertices have preimages in Γ_0, they will have different images in Δ. Let x and y be preimages of z in Γ_0 of different types. In order that both x and y be mapped onto z there must be a sequence

$$x = x_0, x_1, x_2, ..., x_n = y$$

of vertices of Γ_0 such that for every i x_i and x_{i+1} have the same image in either Γ_1 or Γ_2 (and all are mapped to z in Δ). If any x_i has degree 3 z already is the image of vertices of degree 3 in both Γ_1, and Γ_2. Thus, all x_i have degree 2 and must have the same type as x or y. But then these must also exist on i such that x_i and x_{i+1} are of different types and their common image (in either Γ_1 or Γ_2) is of degree 3. □

We continue with two remarks.

Remark 1. Suppose Γ is a core graph of a group U and that all vertices of Γ have degree 2 or 3. Then Γ has exactly

$$2r_U - 2$$

vertices of degree 3.

Remark 2. The mappings α_1 and α_2 must be vertex-surjective if for any free factors U^* and V^* of U and V, resp., the equality

$$U \cap V = U^* \cap V^*$$

implies $U^* = U$ and $V^* = V$.

For, if say α_1, is not vertex-surjective. Then α_1 cannot be edge surjective. Suppose y is an edge not in $\alpha_1\Gamma_0$ and T a tree of $\Gamma_1\backslash y$. Then the $\Pi_1(\Gamma_1\backslash y)$ is a proper free factor of U satisfying $U \cap V = U^* \cap V$.

Proof of (5). Let S be the set of vertices of degree 3 in Δ. By our assumption, Remark 2 and the Lemma we can partition S into three sets: A set S_0 of k_0 vertices, every one of which is the image of degree 3 vertices in both Γ_1 and Γ_2, a set S_1, of k_1 vertices, non of which has a degree 3 preimage in Γ_1, and a set S_2 of k_2 vertices, non of which has a degree 3 preimage in Γ_2.

Furthermore, let $m_z(n_z)$ be the number of degree 3 vertices of $\Gamma_1(\Gamma_2)$ mapped into $z \in \Delta$. Finally, let N be the number of degree 3 vertices in Γ_0, m be the number of degree 3 vertices in Γ_1 and n the number of those in Γ_2. Clearly

$$N \leq \sum_{z \in S} m_z n_z.$$

We also observe that

$$\begin{array}{rclcl}
\sum_{z\in S} m_z n_z & = & m \cdot n & - & \sum_{z \neq w} m_z n_w \\
& \leq & m \cdot n & - & \sum_{z,w \in S_0} m_z n_w - \sum_{z \in S_1} m_z n_w \\
& \leq & m \cdot n & - & \sum_{z \in S_0} (m - m_z) n_z - m \cdot k_1 \\
& \leq & m \cdot n & - & \sum_{z \in S_0} (m - m_z) - m k_1 \\
& \leq & m \cdot n & - & m k_0 + \sum_{z \in S_0} m_z - m k_1 \\
& \leq & m \cdot n & - & m(k_0 + k_1 - 1)
\end{array}$$

Analogously one shows

$$N \leq m \cdot n - n(k_0 + k_2 - 1).$$

We can assume that $U \cap V$ has at least rank 2. Then $k_0 \geq 1$. Since $k_0 + k_1 + k_2 = 2r_{U \vee V} - 2$ we thus have $k_1 + k_2 \leq 2r_{u \vee V} - 3$ and hence either k_1 or $k_2 \geq r_{U \vee V} - 2$. This implies that either $k_0 + k_1$ or $k_0 + k_2$ or $k_0 + k_2 \geq r_{U \vee V} - 1$. Thus

$$N \leq m \cdot n - \min(n, m) \cdot (r_{U \vee V} - 1),$$

or, equivalently,

$$2r_{U \cap V} - 2 \leq (2r_U - 2)(2r_V - 2) - \min(2r_V - 2, 2r_V - 2)(r_{U \vee V} - 1),$$

and therefore

$$r_{U \vee V} - 1 \leq 2(r_U - 1)(r_V - 1) - (\min\{r_U, r_V\} - 1)(r_{U \vee V} - 1)\Box$$

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